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Lynden-Bell and Tsallis distributions for the HMF model

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Abstract. Systems with long-range interactions can reach a Quasi Stationary State (QSS) as a result of a violent collisionless relaxation. If the system mixes well (ergodicity), the QSS can be predicted by the statistical theory of Lynden-Bell (1967) based on the Vlasov equation. When the initial condition takes only two values, the Lynden-Bell distribution is similar to the Fermi-Dirac statistics. Such distributions have recently been observed in direct numerical simulations of the HMF model (Antoniazzi et al. 2006). In this paper, we determine the caloric curve corresponding to the Lynden-Bell statistics in relation with the HMF model and analyze the dynamical and thermodynamical stability of spatially homogeneous solutions by using two general criteria previously introduced in the literature. We express the critical energy and the critical temperature as a function of a degeneracy parameter fixed by the initial condition. Below these critical values, the homogeneous Lynden-Bell distribution is not a maximum entropy state but an unstable saddle point. Known stability criteria corresponding to the Maxwellian distribution and the water-bag distribution are recovered as particular limits of our study. In addition, we find a critical point below which the homogeneous Lynden-Bell distribution is always stable. We apply these results to the situation considered in Antoniazzi et al. For a given energy, we find a critical initial magnetization above which the homogeneous Lynden-Bell distribution ceases to be a maximum entropy state. For an energy $U = 0.69$, this transition occurs above an initial magnetization $M_x = 0.897$. In that case, the system should reach an inhomogeneous Lynden-Bell distribution (most mixed) or an incompletely mixed state (possibly fitted by a Tsallis distribution). Thus, our theoretical study proves that the dynamics is different for small and large initial magnetizations, in agreement with numerical results of Pluchino et al. (2004). This new dynamical phase transition may reconcile the two communities by showing that they study different regimes.

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1 Introduction

The dynamics and thermodynamics of systems with long-range interactions is a topic of active research [1]. Examples include self-gravitating systems, two-dimensional vortices, neutral and non-neutral plasmas, chemotaxis of bacterial populations, just to mention a few. In addition to these physical systems, a toy model called the Hamiltonian Mean Field (HMF) model is widely studied [2–21] because it displays many features of more realistic systems with long-range interactions, like gravity, while being amenable to a simpler mathematical treatment [12].

The HMF model is known to display two successive types of relaxation, like for stellar systems and two dimensional vortices [22,23]. The first stage of the dynamics is a violent collisionless relaxation leading to a QSS after a few dynamical times. This QSS is in general different from the Boltzmann distribution. The second stage is a slow "collisional" relaxation (due to granularities and finite N effects) leading to the Boltzmann distribution which is the statistical equilibrium state of the system. The "collisional" relaxation time increases algebraically with the number of particles so that the QSS has a very long lifetime which becomes infinite in a proper thermodynamic limit $N \to +\infty$.

The nature of the QSS has created an intense debate in the statistical mechanics community. Two different approaches have been developed. Some authors [6,8,15] inspired by the work of Tsallis [24] have proposed to interprete these QSS in terms of a non-extensive thermodynamics based on the so-called q-entropy which is a generalization of the Boltzmann entropy. Other authors [9,12,25,26] inspired by the work of Lynden-Bell [22,27] in astrophysics, have proposed to interprete these QSS in terms of a statistical mechanics of the Vlasov equation called the theory of violent relaxation[1]. The idea

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[1] This theory is relatively well-known in astrophysics and 2D turbulence [22,27,28] but it took some time to diffuse in the statistical mechanics community despite several efforts of the author to publicize it [22,25,29,30]. In particular, the possibility to apply the Lynden-Bell theory to the HMF model (but also the limitations of its application) was mentioned in several papers [12,25,31].

of Lynden-Bell is to determine the *most probable state* of the system resulting from phase mixing compatible with all the constraints imposed by the Vlasov dynamics. This assumes that the system mixes well so that a hypothesis of ergodicity is made. If the initial distribution takes only two values, Lynden-Bell predicts at meta-equilibrium a coarse-grained distribution similar to the Fermi-Dirac statistics in quantum mechanics [27,28,32]. In a recent paper, Antoniazzi et al. [26] have performed direct numerical simulations of the HMF model and found situations in which the Lynden-Bell distribution provides a good description of the QSS without ad hoc fitting parameter. These numerical results are a good motivation to investigate these distributions in more detail, as done in this paper.

In Section 2, we briefly recall the Lynden-Bell theory of violent relaxation with notations appropriate to the HMF model. A more thorough description of this theory can be found in the classical paper [27] and in [28,30,31]. In Section 3, we determine the equation of state $p(\rho)$ associated with the Lynden-Bell distribution $f(\theta, v)$. We consider two particular limits: the dilute limit where the Lynden-Bell distribution becomes similar to the Maxwell-Boltzmann distribution and the completely degenerate limit where the Lynden-Bell distribution becomes a step function (water-bag) similar to the Fermi distribution in quantum mechanics. In Section 4, we determine the caloric curve $\beta(E)$ corresponding to the Lynden-Bell statistics in relation with the HMF model. We restrict ourselves to spatially homogeneous distributions. In Section 5, we analyze the dynamical and thermodynamical stability of these homogeneous solutions by using two general criteria previously introduced in the literature: one is based on the distribution function of the system [9] and the other on the velocity of sound $c_s^2 = p'(\rho)$ in the corresponding barotropic gas [12]. These criteria are equivalent. We express the critical energy and the critical temperature as a function of a degeneracy parameter μ fixed by the initial condition. The known stability criteria corresponding to the Maxwell distribution and the water-bag distribution [9,12] are recovered as particular cases of our study since these distributions are two limits of the Lynden-Bell distribution. For $E < E_c(\mu)$ or $T < T_c(\mu)$ the homogeneous Lynden-Bell distribution is *not* a maximum entropy state. Therefore, it is not expected to be achieved as a result of violent relaxation. In that case, the system may reach an inhomogeneous Lynden-Bell distribution (if it mixes well), or another distribution (if it does not mix well). A critical point $\mu_* = 0.68786...$ is found below which the homogeneous phase is always stable whatever the value of energy and temperature. In Section 6, we apply our results to the situation considered in Antoniazzi et al. [26]. For a given energy, we find a critical initial magnetization above which the homogeneous Lynden-Bell distribution ceases to be a maximum entropy state and becomes a saddle point. For an energy $U = 0.69$, this transition occurs above an initial magnetization $M_x = 0.897$. For $M_x > 0.897$, the system should reach an inhomogeneous Lynden-Bell distribution (most mixed state) or an incompletely mixed state (possibly spatially homogeneous). Thus, our theoretical study proves that the dynamics is radically different for small and large initial magnetizations. This tends to corroborate the claim of Pulchino et al. [8] who made a similar observation on the basis of numerical simulations. In the Conclusion, we stress the limitations of the Lynden-Bell theory and the possibility that the QSS can be described by other types of distributions when the system does not mix well (incomplete violent relaxation). Indeed, the Vlasov equation admits an infinite number of steady states and the system can be trapped in one of them during the collisionless dynamics [25]. The Tsallis distributions (corresponding to stellar polytropes in astrophysics) are *particular* stationary solutions of the Vlasov equation which can sometimes provide a good fit of the QSS in case of incomplete relaxation [6]. However, there is no fundamental reason why these distributions should always (universally) be selected by the dynamics and, indeed, many other distributions can emerge in case of incomplete relaxation, depending on the initial conditions, on the value of the control parameters, and on the route to equilibrium [25]. The Tsallis distributions form just a one-parameter family of steady states of the Vlasov equation [33] and there is no theoretical justification of using them unless one invokes their simplicity and popularity. Similarly, stellar polytropes in astrophysics represent famous stationary solutions of the Vlasov-Poisson system that can provide simple mathematical models of galaxies or convenient fits of astrophysical systems in certain cases, but other distributions can also be considered [34]. In fact, real galaxies are *not* described by polytropic (or Tsallis) distributions [25,34,35].

2 Theory of violent relaxation for the HMF model

The HMF model is a system of N particles moving on a circle and interacting via a cosine binary potential, e.g. [5,12]. The dynamics of these particles is governed by the Hamilton equations

$$\frac{d\theta_i}{dt} = \frac{\partial H}{\partial v_i}, \qquad \frac{dv_i}{dt} = -\frac{\partial H}{\partial \theta_i},$$

$$H = \frac{1}{2}\sum_{i=1}^{N} v_i^2 - \frac{k}{4\pi}\sum_{i\neq j}\cos(\theta_i - \theta_j). \tag{1}$$

This system has an unusual thermodynamic limit defined by $N \to +\infty$ with $\epsilon = 8\pi E/kM^2$ and $\eta = kM/4\pi T$ fixed (here $M = Nm$ is the total mass and we have taken $m = 1$). We can rescale the parameters of the problem so that the coupling constant scales like $k \sim 1/N$ while $E \sim N$ and $T \sim 1$ [23]. For $N \to +\infty$ in this proper thermodynamic limit, the evolution of the distribution function (DF) is governed by the Vlasov equation [9,12,23]:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial \theta} - \frac{\partial \Phi}{\partial \theta}\frac{\partial f}{\partial v} = 0, \tag{2}$$

$$\Phi(\theta, t) = -\frac{k}{2\pi}\int_0^{2\pi}\cos(\theta - \theta')\rho(\theta', t)d\theta'. \tag{3}$$

Starting from an unstable initial DF $f_0(\theta, v)$, the Vlasov equation coupled to the meanfield potential (3) generates a complicated mixing process at the end of which the *coarse-grained* DF $\overline{f}(\theta, v, t)$ achieves a quasi-stationary state $\overline{f}_{QSS}(\theta, v)$. If the system mixes well[2], the QSS is described by the Lynden-Bell distribution [22,25,27]. If the initial DF takes only two values $f_0 = \eta_0$ and $f = 0$, the QSS predicted by Lynden-Bell is obtained by maximizing the mixing entropy [30]:

$$S_{LB} = -\int \left\{ \frac{\overline{f}}{\eta_0} \ln \frac{\overline{f}}{\eta_0} + \left(1 - \frac{\overline{f}}{\eta_0}\right) \ln \left(1 - \frac{\overline{f}}{\eta_0}\right) \right\} d\theta dv, \tag{4}$$

at fixed mass

$$M = \int \rho d\theta, \tag{5}$$

and energy

$$E = \frac{1}{2} \int f v^2 d\theta dv + \frac{1}{2} \int \rho \Phi d\theta, \tag{6}$$

where $\rho = \int f dv$ is the spatial density. We thus have to solve the optimization problem

$$\mathrm{Max}\{S[\overline{f}] \quad | \quad E[\overline{f}] = E, M[\overline{f}] = M\}. \tag{7}$$

Writing the first order variations as

$$\delta S - \beta \delta E - \alpha \delta M = 0 \tag{8}$$

where $\beta = 1/T$ and α are Lagrange multipliers, one obtains the Lynden-Bell distribution function

$$\overline{f} = \frac{\eta_0}{1 + \lambda e^{\beta(\frac{v^2}{2} + \Phi(\theta))}}, \tag{9}$$

where $\lambda = e^{\alpha} > 0$ plays the role of a fugacity. Morphologically, this distribution function is similar to the Fermi-Dirac statistics [27,28,32] so that we shall find many analogies with quantum mechanics.

The Lynden-Bell functional (4) is an entropy because it is proportional to the logarithm of the disorder, where the disorder is equal to the number of microstates consistent with a given macrostate. Indeed, the Lynden-Bell entropy is obtained from a combinatorial analysis [27,31]. Therefore, its maximization at fixed mass and energy determines the most probable macrostate, i.e. the one that is the most represented at the microscopic (fine-grained) scale. In this sense, the optimization problem (7) is a condition of *thermodynamical stability* for the collisionless relaxation. Alternatively, the Lynden-Bell functional (4) can also be interpreted as a particular H-function in the sense of Tremaine et al. [36]. Indeed, it is of the form

$$S = -\int C(\overline{f}) d\theta dv, \tag{10}$$

where C is convex. In that context, the optimization problem (7) is a condition of formal nonlinear dynamical stability with respect to the Vlasov equation [9,12,36–38]. Therefore, the maximization of S at fixed E and M guarantees that the statistical equilibrium macrostate is stable with respect to the perturbation on the microscopic scale (thermodynamical stability) and that the coarse-grained DF $\overline{f}$ is stable for the Vlasov equation with respect to macroscopic perturbations (nonlinear dynamical stability).

We emphasize that it is only when the initial DF takes two values η_0 and 0 that the Lynden-Bell entropy can be expressed in terms of the coarse-grained DF $\overline{f}$ as in equation (4). In general, the Lynden-Bell entropy is a functional of the probability distribution of phase levels $\rho(\theta, v, \eta)$ of the form:

$$S_{LB}[\rho] = -\int \rho \ln \rho \, d\theta dv d\eta, \tag{11}$$

and the coarse-grained DF is given by $\overline{f} = \int \rho \eta d\eta$ [25,27,28,31]. The general Lynden-Bell distribution is expressed as a superposition of Fermi-Dirac distributions of the form

$$\overline{f} = \frac{\int \chi(\eta) \eta e^{-\eta(\beta\epsilon + \alpha)} d\eta}{\int \chi(\eta) e^{-\eta(\beta\epsilon + \alpha)} d\eta}, \tag{12}$$

where we have noted $\epsilon = v^2/2 + \Phi(\theta)$ the energy per particle. This is similar to a sort of superstatistics [31] where the function $\chi(\eta)$ is determined indirectly by the initial condition. Therefore, the expression (4) of the collisionless entropy is not universal; it is valid only in the two-levels approximation. We note also that the expression of the collisionless entropy given by [14] is not correct. These authors do not introduce the notion of coarse-graining and phase mixing, nor the local distribution of phase levels $\rho(\theta, v, \eta)$, which is capital in the theory of violent relaxation to describe the QSS [27]. The correct form of entropy for the violent relaxation process (based on the Vlasov equation) is the Lynden-Bell entropy (11) as claimed in [25].

It should be stressed that the theory of violent relaxation is valid for many systems with long-range interactions described by the Vlasov equation, not only for the HMF model [23]. Historically, this theory was first introduced in astrophysics for collisionless stellar systems [27]. The calculation of self-gravitating Fermi-Dirac spheres corresponding to the Lynden-Bell distribution was performed in [32] and the caloric curve $\beta(E)$ was obtained as a function of a degeneracy parameter μ. An

[2] Lynden-Bell [27] introduces the notion of *microstates* corresponding to the finely striated structure of the DF and *macrostates* corresponding to the smoothed-out (coarse-grained) DF. Using the standard postulate of statistical mechanics, he assumes that all the accessible microstates (with the right value of the integral constraints) are *equiprobable*. Therefore, if the system "mixes well", it will be found at meta-equilibrium in the macrostate which is represented by the maximal number of microstates. This *most probable* (most mixed) macrostate is obtained by maximizing the Lynden-Bell mixing entropy under all the constraints of the Vlasov equation [31].

equivalent theory of violent relaxation was developed in two-dimensional turbulence described by the 2D Euler equation to account for the structure and robustness of large-scale vortices such as Jupiter's great red spot [39,40]. The analogy between 2D vortices and stellar systems was discussed in [22,28,29]. Many numerical simulations have been performed in the two domains to test the successes and the failures of the Lynden-Bell prediction (see [25] for some references). The limitations of the Lynden-Bell theory will be discussed in the Conclusion. In the following, we shall assume that the QSS is described by the Lynden-Bell distribution (statistical equilibrium state of the Vlasov equation) and we use a presentation similar to that developed in the gravitational context [32].

3 Properties of the Lynden-Bell distribution

In this section, we discuss some properties of the Lynden-Bell distribution (9) and consider two limit forms of this distribution: the Maxwell-Boltzmann distribution obtained in the non-degenerate limit and the Fermi distribution (water-bag) obtained in the completely degenerate limit.

3.1 The equation of state

To any distribution function $\overline{f} = \overline{f}(\epsilon)$ depending only on the energy $\epsilon = v^2/2+\Phi(\theta)$, one can associate a corresponding barotropic equation of state $p(\rho)$ [12]. The density and the pressure are defined by

$$\rho = \int_{-\infty}^{+\infty} \overline{f} dv, \tag{13}$$

$$p = \int_{-\infty}^{+\infty} \overline{f} v^2 dv. \tag{14}$$

Let us determine the equation of state associated with the Lynden-Bell distribution (9). Substituting for $\overline{f}$ from equation (9) to equations (13, 14), introducing the notation $\Lambda(\theta) = \lambda e^{\beta\Phi(\theta)}$, and performing the change of variables $x = \beta v^2/2$, we obtain

$$\rho = \left(\frac{2}{\beta}\right)^{1/2} \eta_0 I_{-1/2}(\Lambda), \tag{15}$$

$$p = \left(\frac{2}{\beta}\right)^{3/2} \eta_0 I_{1/2}(\Lambda), \tag{16}$$

where we have defined the Fermi integrals

$$I_n(t) = \int_0^{+\infty} \frac{x^n}{1+te^x} dx. \tag{17}$$

By eliminating Λ between equations (15) and (16), we see that the equation of state is barotropic, i.e. the pressure is a function $p(\rho)$ of the density. It is equivalent to the equation of state of an ideal Fermi gas in one dimension.

3.2 The dilute limit (Maxwell-Boltzmann distribution)

In the limit $\Lambda \to +\infty$, the Lynden-Bell distribution reduces to the Maxwell-Boltzmann distribution

$$\overline{f} \simeq \frac{\eta_0}{\lambda} e^{-\beta(\frac{v^2}{2}+\Phi(\theta))}. \tag{18}$$

Since $\overline{f} \ll \eta_0$, this corresponds to a dilute limit (or to a non degenerate limit if we use the terminology of quantum mechanics). In this limit, the Lynden-Bell entropy (4) takes a form similar to the Boltzmann entropy

$$S_{LB} \simeq -\int \frac{\overline{f}}{\eta_0} \ln \frac{\overline{f}}{\eta_0} d\theta dv. \tag{19}$$

The corresponding equation of state is that of an isothermal gas

$$p = \rho T. \tag{20}$$

This result can be obtained directly from equations (15, 16) by using the asymptotic expression of the Fermi-Dirac integrals for $t \to +\infty$:

$$I_n(t) \sim \frac{\Gamma(n+1)}{t}, \qquad (n > -1). \tag{21}$$

3.3 The degenerate limit (water-bag distribution)

In the limit $\Lambda \to 0$, the Lynden-Bell distribution (9) reduces to the Heaviside function

$$\overline{f} = \begin{cases} \eta_0 & (v < v_F), \\ 0 & (v \geq v_F), \end{cases} \tag{22}$$

where

$$v_F(\theta) = \sqrt{-(2/\beta)\ln \Lambda(\theta)}, \tag{23}$$

is a maximum velocity. The distribution (22) is often called the water-bag distribution. It is also similar to the Fermi distribution in quantum mechanics and v_F is similar to the Fermi velocity. Thus, the limit $\Lambda \to 0$ corresponds to a completely degenerate limit in the quantum mechanics terminology.

Using equations (13, 14), the density is given by $\rho = 2\eta_0 v_F$ and the pressure by $p = (2/3)\eta_0 v_F^3$. Eliminating v_F between these two expressions, we find that the equation of state is

$$p = \frac{1}{12\eta_0^2}\rho^3. \tag{24}$$

This is similar to the equation of state of a polytrope $p = K\rho^{1+1/n}$ with an index $n = 1/2$ and a polytropic constant $K = 1/(12\eta_0^2)$. Polytropic distributions (related to Tsallis distributions) have been studied in [12] in relation with the HMF model. The equation of state (24) can also be obtained directly from equations (15, 16) by using the asymptotic expression of the Fermi-Dirac integrals for $t \to 0$:

$$I_n(t) \sim \frac{(-\ln t)^{n+1}}{n+1}, \qquad (n > -1). \tag{25}$$

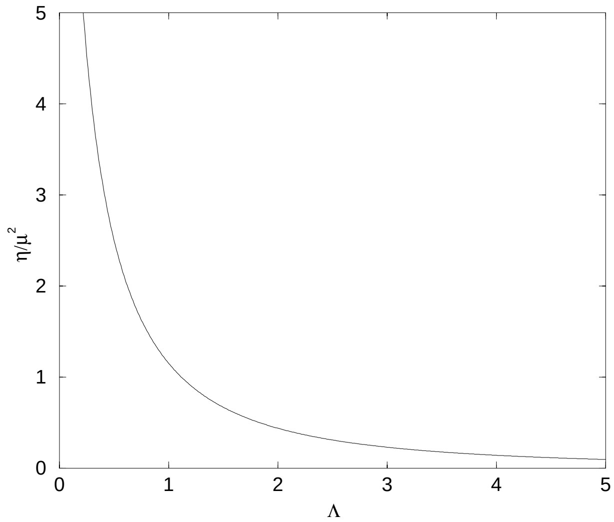

Fig. 1. Inverse temperature η as a function of the fugacity Λ.

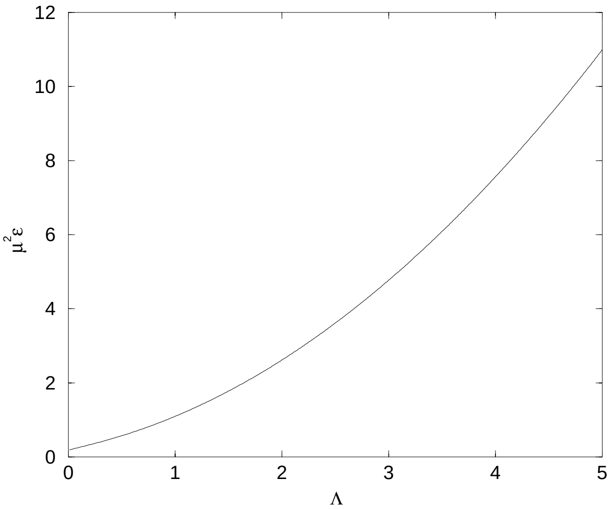

Fig. 2. Energy ϵ as a function of the fugacity Λ.

4 The caloric curve

From now on, we restrict ourselves to spatially homogeneous systems ($\Phi = 0$) so that the Lynden-Bell distribution becomes

$$\overline{f} = \frac{\eta_0}{1 + \lambda e^{\beta \frac{v^2}{2}}}. \tag{26}$$

In this section, we shall determine the relation between the temperature T and the energy E. This defines the caloric curve $T(E)$. Note that the temperature T is a Lagrange multiplier associated with the conservation of energy in the variational problem (8). It also has the interpretation of a kinetic temperature in the Fermi-Dirac distribution (9).

It is useful to introduce dimensionless quantities as in [12]. We define the dimensionless inverse temperature by

$$\eta = \frac{kM}{4\pi T}. \tag{27}$$

We also introduce the degeneracy parameter

$$\mu = \eta_0 \left(\frac{2\pi k}{M}\right)^{1/2}. \tag{28}$$

These notations are similar to those used in the astrophysical context [32]. Using equation (15) and $\rho = M/(2\pi)$ for a homogeneous system, we find that the parameter Λ (fugacity) is related to the temperature and to the degeneracy parameter by

$$\eta = \mu^2 I_{-1/2}(\Lambda)^2. \tag{29}$$

The curve $\eta(\Lambda)$ is decreasing. It behaves as $\eta \sim -4\mu^2 \ln \Lambda$ for $\Lambda \to 0$ and as $\eta \sim \pi\mu^2/\Lambda^2$ for $\Lambda \to +\infty$ (see Fig. 1).

For a homogeneous system, the energy is simply the kinetic energy

$$E = \frac{1}{2}\int \overline{f} v^2 dv d\theta. \tag{30}$$

In terms of the pressure (14), this can be written

$$E = \pi p. \tag{31}$$

Introducing the dimensionless energy [12]:

$$\epsilon = \frac{8\pi E}{kM^2}, \tag{32}$$

and using equation (16), we get

$$\epsilon = \frac{2\mu}{\eta^{3/2}} I_{1/2}(\Lambda). \tag{33}$$

Using equation (29), the foregoing equation can be rewritten

$$\epsilon = \frac{2}{\mu^2}\frac{I_{1/2}(\Lambda)}{I_{-1/2}(\Lambda)^3}. \tag{34}$$

The function $\epsilon(\Lambda)$ increases (see Fig. 2). It starts from $\epsilon(0) = 1/(6\mu^2)$ and increases like $\epsilon(\Lambda) \sim \Lambda^2/(\pi\mu^2)$ for $\Lambda \to +\infty$. Therefore,

$$\epsilon \geq \epsilon_{min} = \frac{1}{6\mu^2}. \tag{35}$$

This minimum energy is similar to the ground state of a one dimensional Fermi gas. In terms of dimensional variables it is given by

$$E_{min} = \frac{M^3}{96\pi^2\eta_0^2}. \tag{36}$$

The caloric curve $\beta(E)$, or equivalently $\eta(\epsilon)$, is obtained by eliminating Λ between equations (29) and (34). We note that the relation between η/μ^2 and $\epsilon\mu^2$ is independent on the degeneracy parameter μ. For $\Lambda \to +\infty$, we recover the relation

$$\eta = \frac{1}{\epsilon}, \qquad (\epsilon \to +\infty), \tag{37}$$

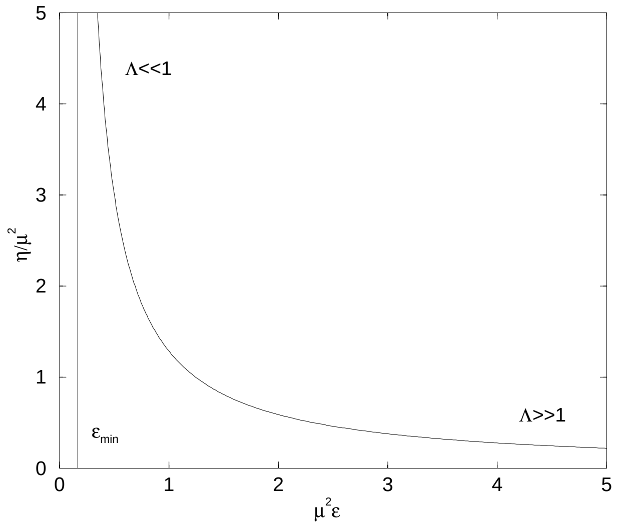

Fig. 3. Caloric curve corresponding to the spatially homogeneous Lynden-Bell distribution.

valid for a classical isothermal gas described by the Maxwell-Boltzmann distribution (18) [12]. In terms of dimensional variables, the relation (37) can be written

$$E = \frac{1}{2}MT, \qquad (E \to +\infty). \tag{38}$$

To investigate the behaviour of the caloric curve close to the ground state, we use the Sommerfeld expansion of the Fermi integrals for $t \to 0$:

$$I_{1/2}(t) = \frac{2}{3}(-\ln t)^{3/2}\left(1 + \frac{\pi^2}{8}(-\ln t)^{-2} + ...\right), \tag{39}$$

$$I_{-1/2}(t) = 2(-\ln t)^{1/2}\left(1 - \frac{\pi^2}{24}(-\ln t)^{-2} + ...\right). \tag{40}$$

Combining these results with equations (29–34), we obtain

$$\eta = \left(\frac{2}{3}\right)^{1/2} \pi\mu(\epsilon - \epsilon_{min})^{-1/2}, \qquad (\epsilon \to \epsilon_{min}). \tag{41}$$

In terms of dimensional variables, this relation can be written

$$E = E_{min}\left[1 + \frac{\pi^2}{36}\left(\frac{MT}{E_{min}}\right)^2 + ...\right]. \tag{42}$$

We note that the energy does not vanish for $T = 0$. This is similar to the effect of a quantum pressure in quantum mechanics, i.e. the distribution function (26) is *not* a Dirac peak $M\delta(v)$ for $T = 0$.

The caloric curve $\eta(\epsilon)$ is represented in Figure 3. It is parameterized by Λ. We note that the dilute limit $\Lambda \to +\infty$ corresponds to $\epsilon \to +\infty$ and the degenerate limit $\Lambda \to 0$ corresponds to $\epsilon \to \epsilon_{min}$.

5 Stability of the homogeneous phase

We have seen that the optimization problem (7) provides a condition of thermodynamical stability (in Lynden-Bell's sense) and a condition of nonlinear dynamical stability with respect to the Vlasov equation. We thus have to select the *maximum* of S at fixed E, M. Indeed, a saddle point of S is unstable and cannot be obtained as a result of a violent relaxation. It can be shown that, for the HMF model, the optimization problem (7) is equivalent to the optimization problem

$$\text{Min}\{F[\overline{f}] = E[\overline{f}] - TS[\overline{f}] \quad | \quad M[\overline{f}] = M\}, \tag{43}$$

where F can be interpreted as a free energy. The criterion (7) can be viewed as a criterion of microcanonical stability and the criterion (43) as a criterion of canonical stability. For the HMF model, the statistical ensembles (microcanonical and canonical) are equivalent so that all the stable solutions can be constructed from the simpler optimization problem (43); no stable solution is forgotten if we solve (43) instead of (7). Now, the optimization problem (43) has been studied in [9,12] for general functionals of the form (10) and a simple stability criterion has been obtained in the case where the steady state is spatially homogeneous. The stability criterion can be expressed either in terms of the distribution function [9] or in terms of the velocity of sound in the corresponding barotropic gas [12]. In this section, we apply these criteria to the Lynden-Bell distribution (26).

5.1 Criterion based on the velocity of sound

It is shown in [12] that the stability criterion (43) for a spatially homogeneous solution of the Vlasov equation can be put in the form of a condition on the velocity of sound $c_s^2 = p'(\rho)$ in the corresponding barotropic gas. A spatially homogeneous distribution is stable with respect to the Vlasov equation (in the above sense) if, and only, if

$$c_s^2 \geq \frac{kM}{4\pi}. \tag{44}$$

This stability criterion exploits the subtle correspondence between a kinetic system described by a DF $f = f(\epsilon)$ and a barotropic gas with an equation of state $p(\rho)$. This correspondence is related to the Antonov first law in astrophysics (see [12,38] for details).

Let us now apply this criterion to the Lynden-Bell distribution. From equations (15, 16), we get

$$p'(\rho) = \frac{2}{\beta}\frac{I'_{1/2}(\Lambda)}{I'_{-1/2}(\Lambda)}. \tag{45}$$

Now, using the identity

$$I'_n(t) = -\frac{n}{t}I_{n-1}(t), \qquad (n > 0) \tag{46}$$

we obtain

$$c_s^2 = \frac{1}{\Lambda\beta}\frac{I_{-1/2}(\Lambda)}{|I'_{-1/2}(\Lambda)|}. \tag{47}$$

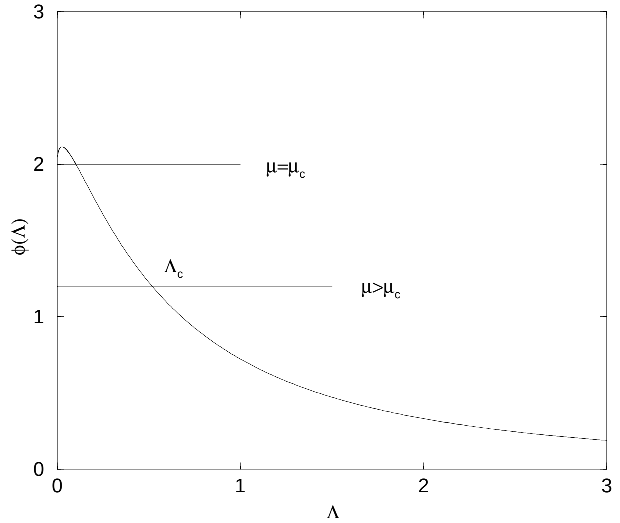

Fig. 4. Graphical construction determining the critical value of the fugacity Λ_c below which the homogeneous Lynden-Bell distribution is not a maximum entropy state anymore. For $\mu > \mu_c$ there is only one intersection.

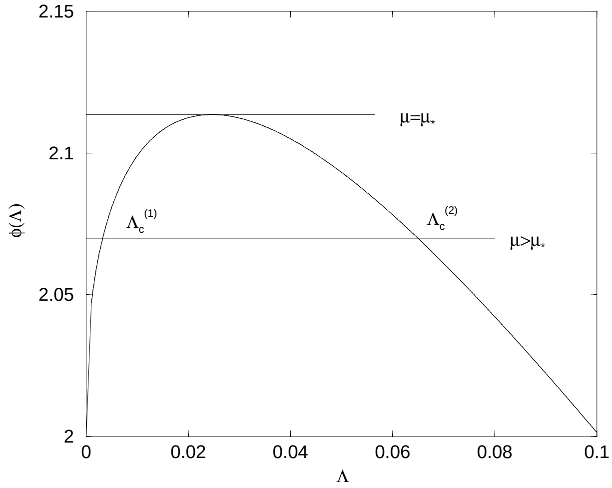

Fig. 5. Graphical construction determining the critical values of the fugacity Λ_c at which the homogeneous Lynden-Bell distribution ceases to be a maximum entropy state. For $\mu_* < \mu < \mu_c$ there are two intersections. The homogeneous distribution is stable (entropy maximum) for $\Lambda < \Lambda_c^{(1)}$ and $\Lambda > \Lambda_c^{(2)}$.

In terms of the dimensionless parameters, the stability criterion (44) can be written

$$\eta \frac{\Lambda |I'_{-1/2}(\Lambda)|}{I_{-1/2}(\Lambda)} \le 1, \tag{48}$$

where Λ is given by

$$I_{-1/2}(\Lambda) = \frac{\sqrt{\eta}}{\mu}, \tag{49}$$

according to equation (29). Combining equations (48, 49), we can rewrite the stability criterion in the form

$$\phi(\Lambda) \equiv I_{-1/2}(\Lambda)\Lambda |I'_{-1/2}(\Lambda)| \le \frac{1}{\mu^2}. \tag{50}$$

The function $\phi(\Lambda)$ starts from $\phi(0) = 2$. It first increases like $\phi(\Lambda) = 2 + (\pi^2/6)(-\ln\Lambda)^{-2}$ for $\Lambda \to 0$, reaches a maximum at $(\Lambda_* = 0.024, \phi_* = 2.1135)$ and then decreases like $\phi(\Lambda) \sim \pi/\Lambda^2$ for $\Lambda \to +\infty$ (see Figs. 4, 5). Therefore, there exists a critical point in the problem. If

$$\mu \le \mu_* \equiv \frac{1}{\sqrt{\phi_*}} = 0.68786, \tag{51}$$

the homogeneous system is stable for any temperature and any energy. In terms of dimensional quantities, this corresponds to

$$\eta_0 \le \mu_* \left(\frac{M}{2\pi k}\right)^{1/2}. \tag{52}$$

Alternatively, for $\mu > \mu_*$ the homogeneous Lynden-Bell distribution is not always a maximum entropy state.

For $\mu > \mu_c$, where

$$\mu_c = \frac{1}{\sqrt{2}} = 0.70710, \tag{53}$$

the equation $\phi(\Lambda) = 1/\mu^2$ has only one solution denoted Λ_c (see Fig. 4). The condition of stability of the homogeneous phase corresponds to $\Lambda > \Lambda_c$. In terms of the temperature (29) or the energy (34), the condition of stability of the homogeneous phase can be written

$$\eta \le \eta_c(\mu), \qquad \epsilon \ge \epsilon_c(\mu), \tag{54}$$

where the critical temperature and the critical energy are defined by the parametric equations

$$\Lambda_c I_{-1/2}(\Lambda_c)|I'_{-1/2}(\Lambda_c)| = \frac{1}{\mu^2}, \tag{55}$$

$$\eta_c = \frac{I_{-1/2}(\Lambda_c)}{\Lambda_c |I'_{-1/2}(\Lambda_c)|}, \tag{56}$$

$$\epsilon_c = \frac{2\Lambda_c |I'_{-1/2}(\Lambda_c)|}{I_{-1/2}(\Lambda_c)^2} I_{1/2}(\Lambda_c), \tag{57}$$

where we recall that

$$I_{-1/2}(t) = \int_0^{+\infty} \frac{1}{\sqrt{x}(1+te^x)} dx, \tag{58}$$

$$I'_{-1/2}(t) = -\int_0^{+\infty} \frac{e^x}{\sqrt{x}(1+te^x)^2} dx, \tag{59}$$

$$I_{1/2}(t) = \int_0^{+\infty} \frac{\sqrt{x}}{1+te^x} dx. \tag{60}$$

For $\mu_* < \mu < \mu_c$, the equation $\phi(\Lambda) = 1/\mu^2$ has two solutions denoted $\Lambda_c^{(1)}$ and $\Lambda_c^{(2)}$ (see Fig. 5). The homogeneous phase is stable for $\Lambda < \Lambda_c^{(1)}$ and for $\Lambda > \Lambda_c^{(2)}$. In

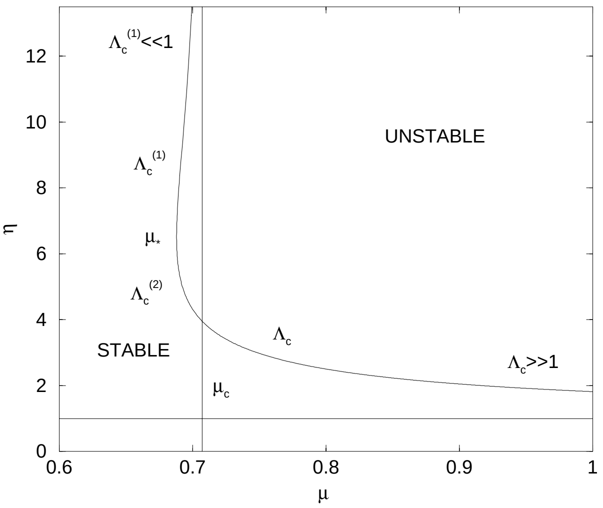

Fig. 6. Stability diagram of the homogeneous phase in the (μ, η) plane.

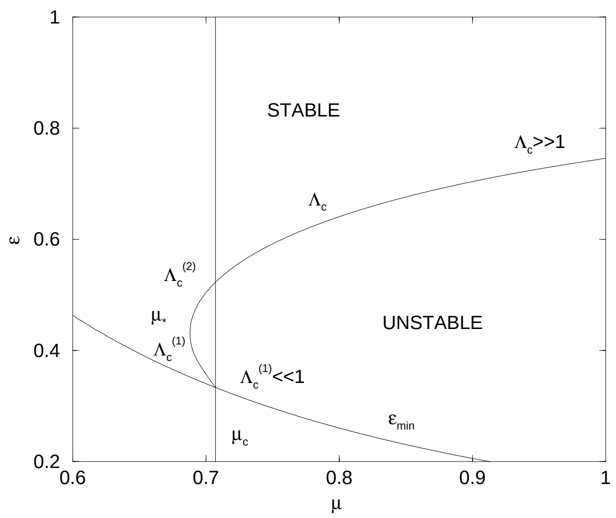

Fig. 7. Stability diagram of the homogeneous phase in the (μ, ϵ) plane.

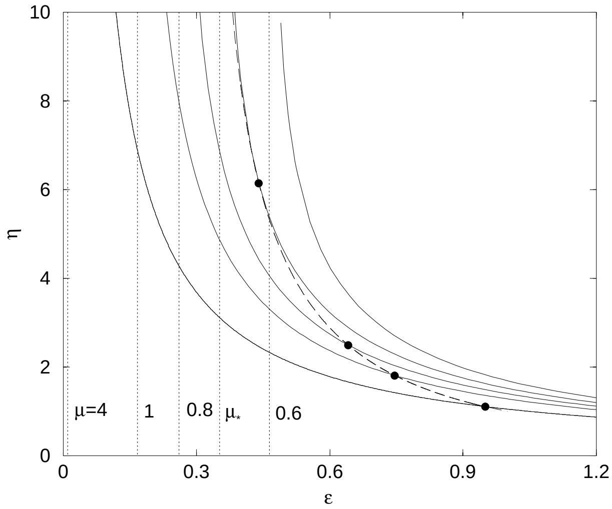

Fig. 8. Caloric curve (29)–(34) corresponding to the homogeneous Lynden-Bell distribution for different values of the degeneracy parameter μ. We have indicated the point (ϵ_c, η_c) at which the series of equilibria becomes unstable. These points are related to each other by the dashed line $(\epsilon_c(\mu) - \eta_c(\mu))$ parameterized by μ. It is obtained from equations (56, 57). For $\mu > \mu_c$ there is only one intersection between the caloric curve and the dashed line. For $\mu_* < \mu < \mu_c$ there are two intersections (see Fig. 9). For $\mu < \mu_*$ there is no intersection and the homogeneous phase is always stable.

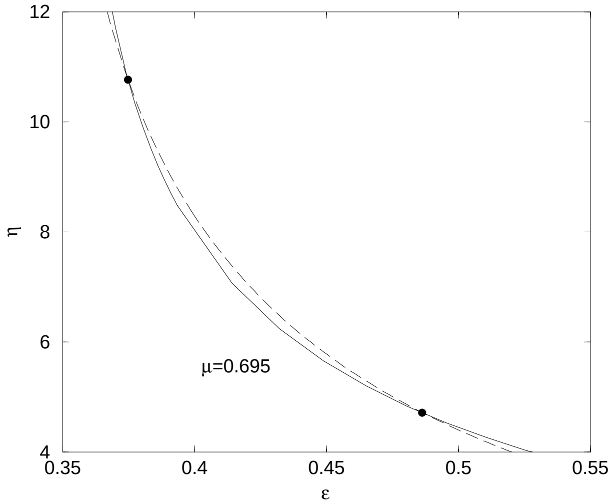

Fig. 9. Enlargement of the previous diagram to show the particularity of the interval $\mu_* < \mu < \mu_c$. When μ lies in this interval (specifically we have taken $\mu = 0.695$) there exists two zones of stability in the series of equilibria separated by a zone of instability.

terms of the temperature, the condition of stability of the homogeneous distribution can be written

$$\eta \leq \eta_c^{(2)}(\mu) \qquad \text{or} \qquad \eta \geq \eta_c^{(1)}(\mu), \qquad (61)$$

where $\eta_c^{(1)}(\mu)$ and $\eta_c^{(2)}(\mu)$ are given by equation (56) with $\Lambda_c^{(1)}$ and $\Lambda_c^{(2)}$ respectively. In terms of the energy, the condition of stability of the homogeneous distribution can be written

$$\epsilon \geq \epsilon_c^{(2)}(\mu) \qquad \text{or} \qquad \epsilon_{min}(\mu) \leq \epsilon \leq \epsilon_c^{(1)}(\mu), \qquad (62)$$

where $\epsilon_c^{(1)}(\mu)$ and $\epsilon_c^{(2)}(\mu)$ are given by equation (56) with $\Lambda_c^{(1)}$ and $\Lambda_c^{(2)}$ respectively.

The stability diagram of the homogeneous Lynden-Bell distribution (26) is plotted in Figures 6, 7 in the (μ, η) plane and in the (μ, ϵ) plane respectively. The representative curve $\eta_c(\mu)$ or $\epsilon_c(\mu)$ marks the separation between the stable (maximum entropy states) and the unstable (saddles point of entropy) regions. We have also plotted the minimum accessible energy $\epsilon_{min}(\mu)$. In Figures 8 and 9, we have represented the caloric curve $\eta(\epsilon)$ for different values of the degeneracy parameter and we have indicated the point at which the series of equilibria becomes unstable (for sufficiently small values of ϵ or sufficiently large values of η). Here, the term "unstable" means that the homogeneous Lynden-Bell distribution is not a maximum entropy state, i.e. (i) it is not the most mixed state (ii) it is dynamically unstable with respect to the Vlasov equation. As a result, it should not be reached as a result of violent relaxation. One possibility is that the system converges to the spatially *inhomogeneous* Lynden-Bell distribution (9) with $\Phi \neq 0$ which is the maximum entropy state (most mixed) in that case (see Sect. 6). Another possibility, always to consider, is that the system does not converge towards the maximum entropy state, i.e. the relaxation is *incomplete* (see Sect. 6 and the Conclusion).

5.2 Limit cases

In the non-degenerate limit $\Lambda_c \to +\infty$, using the asymptotic expansion (21), we find that

$$\eta_c \to 1, \qquad \epsilon_c \to 1, \qquad \mu \sim \Lambda_c/\sqrt{\pi}. \tag{63}$$

These results are valid for $\mu \to +\infty$. In terms of dimensional variables, the condition of stability can be rewritten

$$T \geq \frac{kM}{4\pi} \equiv T_c, \qquad E \geq \frac{kM^2}{8\pi} \equiv E_c. \tag{64}$$

This returns the well-known nonlinear dynamical stability criterion (with respect to the Vlasov equation) of a homogeneous system with Maxwellian distribution function (see, e.g., [9,12]). Indeed, for the equation of state (20), the velocity of sound is $c_s^2 = T$ and the stability criterion (44) directly leads to equation (64). This also coincides with the ordinary thermodynamical stability criterion applying to the *collisional* regime, for $t \to +\infty$, where the statistical equilibrium state is the Boltzmann distribution for f (without the bar!).

In the completely degenerate limit $\Lambda_c \to 0$, using the asymptotic expansion (25), we find that

$$\eta_c \to +\infty, \qquad \epsilon_c \to \frac{1}{3}, \qquad \mu \to \frac{1}{\sqrt{2}}. \tag{65}$$

This can be interpreted in terms of the water-bag model described by the DF (22). For the equation of state (24), the velocity of sound is $c_s^2 = \rho^2/(4\eta_0^2) = M^2/(16\pi^2\eta_0^2)$ and the stability criterion (44) gives

$$\eta_0 \leq \left(\frac{M}{4\pi k}\right)^{1/2}, \qquad \text{i.e.} \qquad \mu \leq \frac{1}{\sqrt{2}}. \tag{66}$$

Note, parenthetically, that this criterion can also be written as a condition on the Fermi velocity [12]: $v_F^2 \geq kM/(4\pi)$ since $c_s = v_F$ according to the relations of Sect. 3.3. For any value of μ, the Fermi distribution is valid for the energy $\epsilon = \epsilon_{min} = 1/(6\mu^2)$ corresponding to $\Lambda \to 0$ (ground state). According to the criterion (66), it is stable only for $\mu \leq \mu_c = 1/\sqrt{2}$, i.e

$$\epsilon_{min} \geq \frac{1}{3}. \tag{67}$$

This returns the well-known stability criterion for the water-bag model (see, e.g., [9,12]).

5.3 Criterion based on the distribution function

As shown in [9], the stability criterion associated with the optimization problem (43) can be written in terms of the distribution function as

$$1 + \frac{k}{2}\int_{-\infty}^{+\infty} \frac{f'(v)}{v} dv \leq 0. \tag{68}$$

The equivalence with the criterion (44) is proved in [12]. The criterion (68) can also be obtained by investigating the *linear* dynamical stability of a homogeneous solution of the Vlasov equation [3,7,12]. Substituting equation (26) in equation (68), we find that a homogeneous system described by the Lynden-Bell distribution is stable if and only if

$$1 - k\eta_0\Lambda\left(\frac{\beta}{2}\right)^{1/2}\int_0^{+\infty} \frac{e^x}{\sqrt{x}(1+\Lambda e^x)^2} dx \geq 0. \tag{69}$$

This condition is equivalent to equation (48), as it should, so that the previous stability analysis could have been performed without modification by starting directly from equation (68).

We may note at this place that the notations introduced in this paper and in [12] differ from those usually introduced in the HMF literature, e.g. [9]. This is because we tried to draw a close parallel with the notations introduced in astrophysics, e.g. [41]. However, it is not difficult to find the relation between the two sets of parameters. In particular, we have

$$\epsilon = 4\left(U - \frac{1}{2}\right), \qquad \eta = \frac{\beta}{2}, \tag{70}$$

where U and β are the energy and the inverse temperature used in [9] (note that we prefer using dimensionless parameters constructed with all the dimensional quantities of the problem instead of fixing some of them to specific values). Therefore, our critical values $(\epsilon_c, \eta_c) = (1, 1)$ for the Maxwell distribution correspond to $(U_c, \beta_c) = (3/4, 2)$ and our critical energy $\epsilon_c = 1/3$ for the water-bag distribution corresponds to $U_c = 7/12$, in agreement with [9]. Note furthermore that, within our notations, the stability criterion for polytropes (Tsallis distributions) takes a very neat form (see Eq. (156) in [12]).

6 Application

For illustration, let us apply our results to the numerical study of [26]. We consider an initial condition made of a patch of uniform distribution function $f_0 = \eta_0$ in the interval $(-\pi\Delta\theta \leq \theta \leq \pi\Delta\theta,\ -\Delta v \leq v \leq \Delta v)$ and $f_0 = 0$ outside (water-bag). The density is $\rho = 2\eta_0\Delta v$ and the total mass is

$$M = \eta_0 4\pi\Delta v\Delta\theta. \tag{71}$$

The energy of a distribution which is symmetric with respect to $\theta = 0$ is given by

$$E = \frac{1}{2}\int f v^2 d\theta dv - \frac{\pi B^2}{k}, \quad (72)$$

where

$$B = -\frac{k}{2\pi}\int_0^{2\pi} \rho(\theta)\cos\theta d\theta, \quad (73)$$

is a parameter similar to the magnetization in spin systems [12]. For the water-bag initial condition f_0, the kinetic energy is given by

$$K_0 = \frac{2}{3}\pi\eta_0 \Delta\theta(\Delta v)^3, \quad (74)$$

and the magnetization by

$$B_0 = -\frac{2k}{\pi}\eta_0 \Delta v \sin(\pi\Delta\theta). \quad (75)$$

It is convenient to introduce the dimensionless parameters

$$x = \pi\Delta\theta, \quad y = \Delta v\left(\frac{8}{\pi k M}\right)^{1/2}, \quad b = -\frac{2\pi B_0}{kM}. \quad (76)$$

Then, the dimensionless initial magnetization can be written

$$b = \frac{\sin x}{x}, \quad (77)$$

the dimensionless energy

$$\epsilon = \frac{\pi^2}{6}y^2 - 2b^2, \quad (78)$$

and the degeneracy parameter

$$\mu = \frac{1}{xy}. \quad (79)$$

For a given energy ϵ, the previous relations relate the initial magnetization b to the degeneracy parameter μ. We note this function $b_\epsilon(\mu)$. It is represented in Figure 10 for a particular value of the energy (see below). According to the stability diagram of Figure 7, the homogeneous Lynden-Bell distribution is stable for any value of the degeneracy parameter if $\epsilon \geq 1$. On the other hand, it is always unstable (i.e., it is not a maximum entropy state) if $\epsilon < 1/3$. Finally, if $1/3 \leq \epsilon \leq 1$, the homogeneous phase is stable only for $\mu_{min}(\epsilon) \leq \mu \leq \mu_{crit}(\epsilon)$ where $\mu_{min}(\epsilon) = 1/\sqrt{6\epsilon}$. Using the curve $b_\epsilon(\mu)$, we can express this stability criterion in terms of the initial magnetization. We first note that $\Delta\theta \in [0, 1]$ so that $0 \leq x \leq \pi$. On the other hand, for $x = \pi$, we find that $b = 0$ and $\epsilon = (\pi y)^2/6$ leading to $\mu = 1/\sqrt{6\epsilon} = \mu_{min}(\epsilon)$. Therefore, if $1/3 \leq \epsilon \leq 1$, the homogeneous phase is stable only for $0 \leq b \leq b_{crit}(\epsilon)$ where $b_{crit}(\epsilon) = b_\epsilon(\mu_{crit})$. Above the critical magnetization $b_{crit}(\epsilon)$, the homogeneous Lynden-Bell distribution is not a maximum entropy state (it is a saddle point).

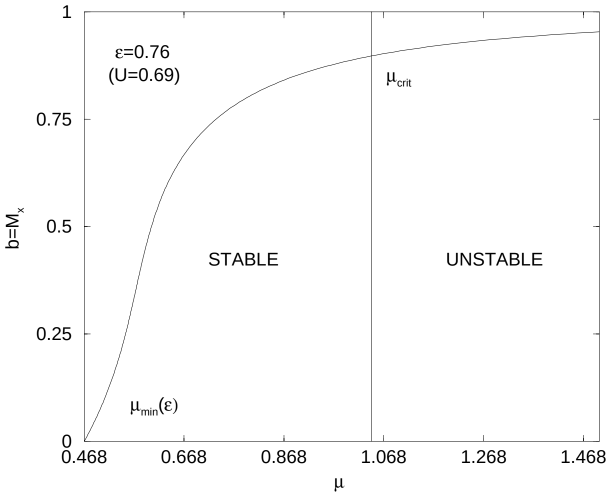

Fig. 10. Initial magnetization $b = M_x$ as a function of the degeneracy parameter μ for a given value of the energy. There exists a critical magnetization, corresponding to $\mu_{crit}(\epsilon)$, above which the homogeneous Lynden-Bell distribution is unstable.

Using the notations of [26], we have

$$\epsilon = 4\left(U - \frac{1}{2}\right), \qquad b = M_x. \quad (80)$$

As in [26], we fix an energy $U = 0.69$. In our notations, it corresponds to $\epsilon = 0.76$. From Figure 7, we find that the critical degeneracy parameter corresponding to this energy is $\mu_{crit} = 1.043...$. The homogeneous phase is stable for $0.468 \leq \mu \leq 1.043$. Using the relation between μ and the initial magnetization $b = M_x$ for $U = 0.69$ represented in Figure 10, we conclude that the homogeneous Lynden-Bell distribution is stable (maximum entropy state) for $M_x \leq 0.897$. Above this critical value, it is not a maximum entropy state so it should not result from a complete violent relaxation: (i) because it is not the most mixed state (ii) because this distribution is dynamically unstable with respect to the Vlasov equation. The existence of a critical magnetization is natural. Indeed, for $M_x \to 1$, the degeneracy parameter $\mu \to +\infty$. Since the Lynden-Bell distribution becomes "non-degenerate" in this limit, the stability criterion is $U > U_c = 3/4$ as for the Maxwell-Boltzmann distribution. Since $U = 0.69 < U_c$ in the simulations, there should exist a critical magnetization $M_{x,crit}$ above which the system becomes unstable. In fact, we already know that the homogeneous Maxwell-Boltzmann distribution is not always a maximum entropy state. Since this is a particular limit of the Lynden-Bell distribution, it is expected that the homogeneous Lynden-Bell distribution itself is not always a maximum entropy state.

If we consider the inhomogeneous Lynden-Bell distribution (9) and assume, without loss of generality, that it is symmetric with respect to $\theta = 0$ (chosen as the origin), the mass, the magnetization and the energy can be expressed as

$$M=\eta_0\left(\frac{2}{\beta}\right)^{1/2}\int_0^{2\pi} I_{-1/2}(\lambda e^{\beta B\cos\theta})\,d\theta, \quad (81)$$

$$B=-\frac{kM}{2\pi}\frac{\int_0^{2\pi} I_{-1/2}(\lambda e^{\beta B\cos\theta})\cos\theta\,d\theta}{\int_0^{2\pi} I_{-1/2}(\lambda e^{\beta B\cos\theta})\,d\theta}, \quad (82)$$

$$E=\frac{1}{2}\eta_0\left(\frac{2}{\beta}\right)^{3/2}\int_0^{2\pi} I_{1/2}(\lambda e^{\beta B\cos\theta})\,d\theta-\frac{\pi B^2}{k}, \quad (83)$$

where we have used $\Phi = B\cos\theta$. Equations (81–83) determine the equilibrium magnetization B as a function of the temperature T or energy E for a given value of the degeneracy parameter μ. The homogeneous state $B = 0$ is always a solution of these equations and we recover equations (29, 34). On the other hand, in the dilute limit $\lambda \to +\infty$, using the expansion (21), equation (82) reduces to the implicit equation (25) of [12] for the Maxwell-Boltzmann distribution. To determine the critical temperature at which the inhomogeneous solution appears, we expand the equations (81, 82) for $B \to 0$ as discussed in [12] at a more general level. This yields

$$B=-\frac{kM}{4\pi}\frac{\lambda I'_{-1/2}(\lambda)}{I_{-1/2}(\lambda)}\beta B\left[1+J(\lambda)(\beta B)^2+...\right], \quad (84)$$

where

$$J(\lambda)=\frac{1}{8}+\frac{3\lambda}{8}\frac{I''}{I'}+\frac{\lambda^2}{8}\frac{I'''}{I'}-\frac{\lambda}{4}\frac{I'}{I}-\frac{\lambda^2}{4}\frac{I''}{I}, \quad (85)$$

and we have noted I for $I_{-1/2}(\lambda)$. It can be shown that $J(\lambda)$ is always negative with $J(0) = 0$ (Fermi) and $J(+\infty) = -1/8$ (Maxwell). Therefore, equation (84) will have a solution $B \neq 0$ if, and only, if

$$-\frac{kM}{4\pi}\frac{\lambda I'_{-1/2}(\lambda)}{I_{-1/2}(\lambda)}\beta > 1. \quad (86)$$

This is precisely the condition (48) giving the point at which the homogeneous phase becomes unstable (i.e. becomes a saddle point of entropy). Therefore, in continuity with the Maxwell-Boltzmann distribution (see, e.g., [12]), the inhomogeneous Lynden-Bell phase appears precisely when the homogeneous Lynden-Bell phase becomes unstable. This result is in fact quite general as discussed in Section 4.3 of [12]. For the Maxwell-Boltzmann distribution, it has been shown in [12] by an explicit calculation (solving an eigenvalue equation associated with the second order variations of entropy) or by using the Poincaré argument for linear series of equilibria that the inhomogeneous distribution, when it exists, is always a maximum entropy state. By continuity, we expect in the present case that the inhomogeneous Lynden-Bell distribution is a maximum entropy state for the functional (4). Therefore, in the situation considered in [26], the homogeneous Lynden-Bell distribution is a maximum entropy state for $M_x < 0.897$ and it becomes an unstable saddle point above this critical magnetization. For $M_x > 0.897$ the maximum entropy state is an inhomogeneous Lynden-Bell distribution.

The existence of a critical initial magnetization is interesting because there has been recent claims, based on numerical simulations, that the dynamics should be different for small ($M0$) and large ($M1$) magnetizations [8]. Our theoretical study gives further support to that claim. For $M_x < 0.897$, our study based on the statistical mechanics of violent relaxation predicts that the system should tend to an homogeneous Lynden-Bell distribution with Gaussian tails. This is confirmed by the numerical work of Antoniazzi et al. [26] who show simulations up to $M_x = 0.7$. In that case, all the results can be explained by standard statistical mechanics and kinetic theory [10,12,16,23,42,43]. However, for $M_x > 0.897$, the situation is different. The statistical mechanics of violent relaxation predicts that the system should tend to an inhomogeneous Lynden-Bell distribution (most mixed). The occurence of a change of regime (dynamical phase transition) seems to be consistent with numerical results of Pluchino et al. [8] who observe the appearance of *structures* in the μ-space for large initial magnetization (but not for small). The fact that $M_{x,crit} = 0.897$ is close to one is also in qualitative agreement with recent reports of Pluchino and Rapisarda [44]. When the system becomes spatially inhomogeneous or involves phase space structures, the statistical mechanics and kinetic theory become complicated and may lead to anomalies as mentioned in [17]. For $M_x > 0.897$, since the dynamics becomes more complex, it is possible that the system will not mix well during violent relaxation so that the inhomogeneous Lynden-Bell distribution (most mixed state) will *not* be achieved in practice. This is in fact what is observed. Indeed, for $M_x = 1$, the degeneracy parameter $\mu \to +\infty$ so that the Lynden-Bell distribution coincides with the Maxwell-Boltzmann distribution (non degenerate limit). Now, for $M_x = 1$ the QSS is *not* a Gaussian as shown in [6]. Thus, we expect that for $M_x > 0.897$ the system will be trapped in an *incompletely mixed* state (which may be spatially homogeneous or weakly inhomogeneous). This is a stable stationary solution of the Vlasov equation but different from the Lynden-Bell distribution due to incomplete relaxation. Numerical simulations [6] show that, in certain cases, this state can be fitted by a Tsallis distribution. The Tsallis distributions form a *particular* one-parameter family of stationary solutions of the Vlasov equation (indexed by q) known as stellar polytropes in astrophysics [33,37]. However, there is absolutely *no fundamental reason* why the Tsallis distributions should be selected in a *universal manner* as a result of incomplete violent relaxation [25,33]. Other fits can work as well or even better (see, e.g., [11]) depending on the situation. In fact, *any* distribution function of the form $f = f(\epsilon)$ where $\epsilon = v^2/2 + \Phi$ is the individual energy ($\Phi = 0$ for a homogeneous system) is a steady state of the Vlasov equation. Furthermore, if $f(\epsilon)$ maximizes an H-function $H = -\int C(f)d\theta dv$ (where C is convex) at fixed mass M and energy E, then it is nonlinearly dynamically stable with respect to the Vlasov equation [12]. Tsallis distributions associated with $C(f) = (1/(q-1))(f^q - f)$ are a special case of distribution functions enjoying these properties but infinitely many other distributions can be considered as well. The *non-universality* of the distribution function of the QSS, depending on the dynamics, is further discussed in the Conclusion and in [25]. Although the Tsallis distributions have not a fundamental justification in the context of incomplete violent relaxation, an interest of these distributions is their *mathematical simplicity* and this is why it is convenient to try this fit first. Such fits have indeed been considered in [6] and the best fit is obtained for $q_* = 7$[3]. It gives a reasonable, but not perfect, agreement with observations. As said previously, there is no fundamental reason why the QSS should be precisely described by a Tsallis distribution, so the deviation between observation and fit should not cause surprise. In our opinion, the detailed structure of the QSS is unpredictable in case of incomplete violent relaxation (see Conclusion and [25]). This unpredictability is already present in the Tsallis q parameter which is a free parameter which has to be adjusted to the situation in order to fit the results at best. More generally, we argue that the pdf of the QSS should not always be a Tsallis distribution, even in case of incomplete relaxation (i.e. when the Lynden-Bell prediction fails). More general distributions could arise [25,33,37]. However, the Tsallis formalism is nice because it provides *simple* analytical expressions of non-standard distributions (with one single parameter q) that can be handled easily (hence its popularity!). Furthermore, an effective kinetic theory, based on generalized Fokker-Planck equations, can be developed in consistency with these distributions [45,46]. Although more general kinetic theories can be developed as well for other equilibrium distributions associated with other forms of "generalized entropies" [33,47], the q-Fokker-Planck equations associated with the Tsallis entropy provide a good basis for practical studies (due, again, to their *simplicity*) and they may be representative of more general situations. As suggested in [23,31,47], these *effective* kinetic theories could be useful precisely when standard kinetic theories break down or become complicated [17] due to the emergence of structures in μ-space, non-ergodicity, memory effects, finite N effects etc. Of course, when standard kinetic theory applies [10,12,16,23,42,43], they are not necessary. A plausible scenario is that standard statistical mechanics (based, however, on Lynden-Bell's approach) applies for $M_x < 0.897$ and that anomalies appear for $M_x > 0.897$ because the evolution is non-ergodic and involves phase-space structures, memory effects etc. It is in this regime that Tsallis effective thermodynamics (or more general approaches [33,47]) could be applied. Therefore, our study tends to reconcile two groups of researchers by suggesting that they study two different dynamics, below [26] and above [8] the critical initial magnetization $M_x = 0.897$ (for $U = 0.69$). This is consistent with our general claim [23,25,31,33,37,47] that the Tsallis formalism can be useful in some situations even if it provides, in our opinion, only an *effective* description of complex systems. Things are more complex than often said and they deserve a detailed and careful discussion.

[3] The nonlinear dynamical stability of the homogeneous Tsallis distributions (polytropes) can be studied using the results of [12] (independently, an equivalent calculation has been made in parallel in [11]). The q_* of [6] is related to our q in [12] by $1-q_* = q-1$ hence $q = 2-q_* = -5$. This corresponds to a polytropic index $n = 1/2+1/(q-1) = 1/3$ or $\gamma = 1+1/n = 4$. Our study in [12] shows that such DF are nonlinearly dynamically stable with respect to the Vlasov equation (they maximize the Tsallis H-function [37] at fixed mass and energy) if $\epsilon > \epsilon_{crit} = 1/\gamma = 1/4$ hence $U > 9/16 = 0.5625$ which is fulfilled for $U = 0.69$. By contrast, the homogeneous Maxwell-Boltzmann distribution ($q = 1$) is stable for $U > 3/4$ which is not fulfilled. Therefore, the homogeneous Tsallis distribution with $q_* = 7$ is stable while the homogeneous Lynden-Bell distribution (equivalent to the Maxwell-Boltzmann distribution in the dilute limit $M_x = 1$, $\mu \to +\infty$) is unstable. Note, finally, that the critical energy $\epsilon_{crit} = 1/\gamma$ obtained in [12] is equivalent to the critical energy $U_{crit} = 3/4 + (q_* - 1)/(2(5 - 3q_*))$ obtained in [11] but expressed in a simpler form.

7 Conclusion

In this paper, we have investigated the stability of the spatially homogeneous Lynden-Bell distribution (26) by determining whether it corresponds to a maximum of the functional (4) at fixed mass and energy. This maximization problem provides a condition of thermodynamical stability for the process of violent relaxation (in which case S is interpreted as an entropy) as well as a condition of nonlinear dynamical stability with respect to the Vlasov equation (in which case S is interpreted as a generalized H-function in the sense of [36]). The thermodynamical stability condition ensures that the system is the most mixed state with respect to microscopic (fine-grained) perturbations. The nonlinear dynamical stability condition ensures that the coarse-grained distribution is robust for the collisionless dynamics against macroscopic perturbations. It is particularly interesting to note that the conditions of thermodynamical (in Lynden-Bell's sense) and nonlinear dynamical stability coincide. We have obtained the expression of the critical energy and critical temperature above which the homogeneous phase is stable as a function of the degeneracy parameter. The known stability criteria corresponding to the Maxwell-Boltzmann distribution and the water-bag distribution are recovered as particular limits of our study. Furthermore, a critical point $\mu_* = 0.68786...$ has been found below which the homogeneous phase is always stable, whatever the value of energy and temperature. When the homogeneous Lynden-Bell distribution is unstable (saddle point of entropy) it cannot be achieved as a result of violent relaxation. In that case, the maximum entropy state is an inhomogeneous Lynden-Bell distribution. For a given value of energy, the transition should occur for a sufficiently large value of the initial magnetization ($M_x > 0.897$ for $U = 0.69$). We have suggested that the relaxation becomes incomplete for $M_x > 0.897$ so that the Lynden-Bell prediction fails (this should be checked numerically but this seems to be the case at least for $M_x = 1$ [6]). In that case, other distributions, that are stable stationary solutions of the Vlasov equation, can emerge. In some situations, the QSS can be fitted by the Tsallis distribution (polytrope). This provide a simple characterisation of the QSS but this fit is not expected to be universal or fundamental. The fact that the Lynden-Bell prediction breaks down is the mark of a lack of ergodicity and a lack of efficient mixing. Dynamical anomalies then appear and the Tsallis formalism could be used in that context (as an *effective* description) as considered in [6]. Thus, having evidenced a new dynamical phase transition, our study may reconcile two groups of researchers by showing that they describe in fact different dynamical processes: for $M_x < 0.897$ the dynamics is ergodic and ordinary statistical mechanics (Lynden-Bell [27]) applies and for $M_x > 0.897$ the dynamics is non-ergodic and *effective* generalized thermodynamics, such as Tsallis [24] approach (or more general [33,47] approaches) could be tried.

We would like to conclude this study by emphasizing the limitations of the Lynden-Bell statistical theory at a general level. First of all, the distribution function (9) is only valid when the initial condition takes two values 0 and η_0. Therefore, it is not expected to apply to any initial condition. For more complex initial conditions, the Lynden-Bell prediction is a superposition of Fermi-Dirac distributions for all the phase levels η. Therefore, depending on the initial conditions, the Lynden-Bell distribution can take a wide diversity of forms given by the general formula (12); this is similar to a sort of superstatistics [31] where $\chi(\eta)$ is fixed by the initial condition. The Fermi-Dirac distribution (9) is just a particular case of this general formula for two levels. It is indeed important to stress that the prediction of Lynden-Bell depends on the *details* of the initial condition, not only on the robust conserved quantities E and M. This is at variance with usual statistical mechanics where only the robust constraints (energy, mass, ...) matter. This is due to the existence of Casimir constraints in the Vlasov dynamics that act as *hidden constraints* [31]. When we consider realistic initial conditions, we enter into complications because: (1) we need to discretize the initial condition into several levels and then relate the Lagrange multipliers $\chi(\eta)$ to the hypervolume $\gamma(\eta)$ of each level [28,31]. This makes the application of the Lynden-Bell theory technically complicated and heavy because it involves a lot of control parameters [48]. (2) we do not always know at which scale we must discretize the initial condition and different discretizations may lead to different results as discussed in [49] (p. 284) and in [50]. (3) in addition, there is a debate to decide whether all the Casimirs are conserved (microscopically) or if certain are altered by non-ideal effects during the dynamics so that a *prior* distribution should be introduced instead (this remark applies particularly to realistic systems such as 2D turbulence [33,51–54]). The simple two-levels situation considered in Antoniazzi et al. [26] is not subject to such difficulties and criticisms.

On the other hand, the approach of Lynden-Bell assumes that the system mixes well so that the hypothesis of *ergodicity*[4] which sustains his statistical theory (maximization of the entropy) is fulfilled. Again, this is not expected to be general. Several cases of *incomplete* violent relaxation have been identified in stellar dynamics and 2D turbulence (see some references in [25]). In such cases, the QSS is not exactly described by the Lynden-Bell distribution but it is nevertheless a stable stationary solution of the Vlasov equation. Therefore, distributions different from (9) can emerge in case of incomplete violent relaxation. This is the case for example in the plasma experiment of Huang and Driscoll [55] in 2D turbulence where it has been observed that the QSS is not perfectly well-described by the Lynden-Bell theory. In particular, the density drops to zero at a finite distance instead of decaying smoothly. Boghosian [56] has interpreted this result in terms of Tsallis non-extensive thermodynamics. Alternatively, the deviation from the Lynden-Bell prediction has been interpreted in [48] as a result of an incomplete violent relaxation (non-ergodicity) and a lack of mixing in the core and the halo of the "vortex". In this interpretation, the QSS is viewed as a particular stable stationary solution of the 2D Euler equation which is not the most mixed state. As we understand, the proposal of Tsallis is to introduce an entropy which can take into account *non-ergodic* behaviours[5]. Indeed, generalized entropies make sense only when the system is non ergodic. If the system mixes well, as in the situations considered in [26], the Lynden-Bell theory is the relevant one [25]. Unfortunately, we do not know a priori whether the system will mix well or not; this depends on the dynamics and on the route to equilibrium [25]. We only know *a posteriori* if the Lynden-Bell prediction has worked or failed (like in, e.g., [55]). The possibility that non-ergodic behaviours could be described by a generalized form of entropy is an attractive idea. However, we do *not* believe that complicated non-ergodic behaviours associated with the process of incomplete violent relaxation can be encapsulated in a simple functional such as the q-entropy proposed by Tsallis. When the system does not mix well we can have a wide variety of QSS. This is because the Vlasov (or 2D Euler) equation admits an infinite number of steady states and the system can be trapped in one of them. A kinetic theory of violent relaxation, as initiated in [28,57], is then necessary to account for incomplete relaxation (see [25]).

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[4] It should be clear that, in this paper, we are talking about the ergodicity with respect to the collisionless mixing in relation with the process of violent relaxation, not the ergodicity with respect to the collisional evolution. Collisional relaxation (due to granularities and finite N effects) is not considered here because it occurs on very long timescales for $N \gg 1$ and does not account for the structure of the QSS [12].

[5] In this respect, we should stress that the proper form of Tsallis entropy for the process of violent relaxation is the one given in [25], expressed in terms of $\rho(\theta, v, \eta)$. In the two-levels approximation, it reduces to a q-Fermi-Dirac type entropy [48]. Within this interpretation, the index q would be a measure of mixing. For $q = 1$ (efficient mixing), we recover the Lynden-Bell theory (see [25]).

In conclusion, for these two reasons: (1) dependence on the detailed structure of the initial conditions (through the Casimirs constraints); (2) incomplete violent relaxation (non ergodicity), the QSS is *not* expected to be described by a "universal" distribution such as (26), or even (12). However, as claimed long ago in [48], the Lynden-Bell theory is the only one to make a *prediction* of the QSS without ad hoc parameter. It is usually observed that the Lynden-Bell prediction provides a *fair* description of the QSS in many cases even if there can exist discrepancies that are more or less pronounced due to incomplete relaxation. Thus, the Lynden-Bell distribution gives the tendency to which state the system *should* tend as a result of mixing. However, during the route to equilibrium, mixing may not be sufficient and the system can be frozen in a stationary solution of the Vlasov equation which is not the most mixed state. This process of incomplete relaxation is beautifully described by Binney and Tremaine [34], pp. 266–267, using an analogy with the structure of the Mississippi river. On the other hand, Tsallis distributions form just a particular one-parameter family of stationary solutions of the Vlasov equation (analogous to stellar polytropes) that can sometimes be used as convenient *fits*, in case of incomplete violent relaxation, due to their simple mathematical expression. However, these fits should not work in a universal manner and other fits can work as well, or even better.

There are still debates and controversies about the fact that the collisionless evolution of the HMF model is adequately described by the Vlasov equation. Indeed, the authors of [6,8,15] describe the QSS in terms of Tsallis generalized thermodynamics based on the N-body system, not in terms of the Lynden-Bell thermodynamics based on the Vlasov equation. The Vlasov description is classical in plasma physics and astrophysics (as well as in point vortex dynamics and 2D turbulence where it has the form of the Euler equation [22]) to describe the evolution of the system during the regime where "collisions" are negligible. In that case, there are no correlations between the particles and the N-body distribution function is a product of N one-body distribution functions. Using this property to close the BBGKY hierarchy steming from the Liouville equation, one obtains the Vlasov equation [23]. For a system with weak long-range interactions, it can be shown that this mean field description is correct in a proper thermodynamic limit $N \to +\infty$[6]. A mathematically rigorous derivation of the Vlasov equation is given by Braun and Hepp [59]. However, justifying the Vlasov equation for the collisionless regime of the HMF model is not the end of the story. As noted in [25], the Vlasov equation coupled to a long-range force can have a very complicated behaviour (this is similar to the Euler equation in 2D turbulence [22]). Therefore, the anomalies (non-ergodicity, phase space structures, ...) observed by [6,8,15] in their N-body simulations would probably persist and be observed by directly solving the Vlasov equation. This would describe the $N \to +\infty$ limit of the model. This comparative study (N-body Hamilton equations versus Vlasov equation) has not yet been done for the HMF model but it would be an interesting step to reconcile different approaches. It would also clearly show whether the "anomalies" reported in [6,8,15] are due to finite N effects or if they persist in the thermodynamic limit $N \to +\infty$.

Note finaly that, since the Lynden-Bell distribution (9) is similar to the Fermi-Dirac statistics in quantum mechanics, the results of this paper also describe the (ordinary) statistical mechanics of a system of N fermions on a ring interacting via a cosine potential; this could be called the *fermionic HMF model*. In that context, the maximum value of the distribution function is $\eta_0 = g/h$ where h is the Planck constant and $g = 2s + 1$ the spin multiplicity of the quantum states. In this quantum context, η_0 is fixed by the Pauli exclusion principle (see the analogous situation for self-gravitating fermions in [60]).

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[6] As shown in [12,16,23], the Vlasov equation also describes the collisional relaxation at order $1/N$. This is because the Lenard-Balescu collision term cancels out for 1D systems (this property is known for a long time in plasma physics [58]). Collisions therefore manifest themselves on a timescale larger than N. For the HMF model, numerical simulations show that the relaxation time scales as $N^{1.7}$ [9].

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